Calculus: Early Transcendentals (3rd Edition)

Briggs et al.

Chapter 1 Functions (2 lectures: §1.3 – §1.4)

§1.3 Inverse, Exponential, and Logarithmic Functions

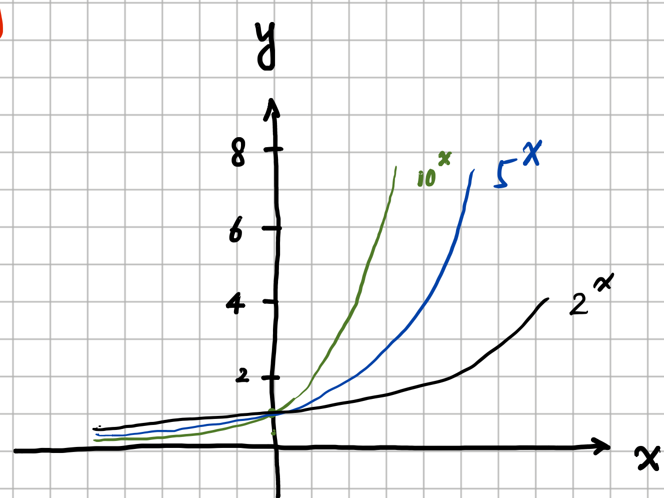
(HW: 3, 5, 12, 17, 18, 22, 23, 29, 33, 49, 53, 59, 78, 81)

- Exponential Functions $f(x) = b^x$, $\{b \in \mathbb{R} : b \neq 1 \text{ and } b > 0\}$

- Exponent Rules $b^x \cdot b^y = b^{x+y}$ $\quad \dfrac{b^x}{b^y} = b^x \cdot b^{-y} = b^{x-y}$

  $(b^x)^y = \underbrace{b^x \cdots b^x}_{y} = b^{xy}$ $\quad b^x > 0$

- Graphs of $y = b^x$ with $b = 2, 5, 10$



- domain of $(f(x) = b^x) = \{x : b > 0, b \neq 1\}$

- $f(x) = b^x \begin{cases} \text{increasing function}, & b > 1 \\ \text{decreasing function}, & b < 1 \end{cases}$

- $e = 2.7182818284 59\cdots$

- **Inverse Functions**

$$x \xrightarrow{f} y \xrightarrow{f^{-1}} x$$

Given $y = f(x)$

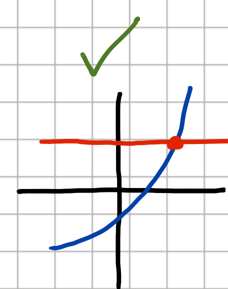
inverse function $x = f^{-1}(y)$ if f^{-1} exists

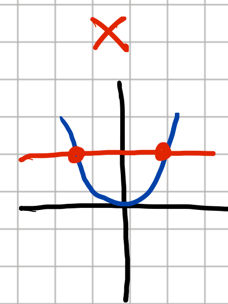
- **One-to-One Functions**

$f(x)$ is a 1-to-1 function on a domain D $\iff$ for all x_1, x_2 in D
$f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2$

- **the horizontal line test**

every horizontal line intersects the graph of a 1-to-1 function at most once

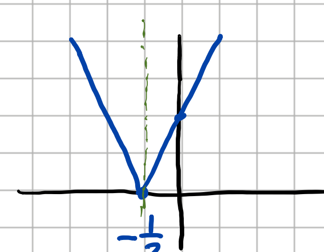




Examples

- Where do inverse exist? (21-26)

#22 $f(x) = |2x+1| = \begin{cases} 2x+1, & 2x+1 > 0 \\ -(2x+1), & 2x+1 < 0 \end{cases}$



inverse of $f(x)$ exists if $x > -\frac{1}{2}$ or $x < -\frac{1}{2}$.

- Finding inverse functions (33-42)

(1) $f(x) = 2x+6$
- let $y = 2x+6$
- solve the equation for x: $x = \frac{y-6}{2}$
- interchange x and y: $y = \frac{x-6}{2} \Rightarrow f^{-1}(x) = \frac{x-6}{2}$

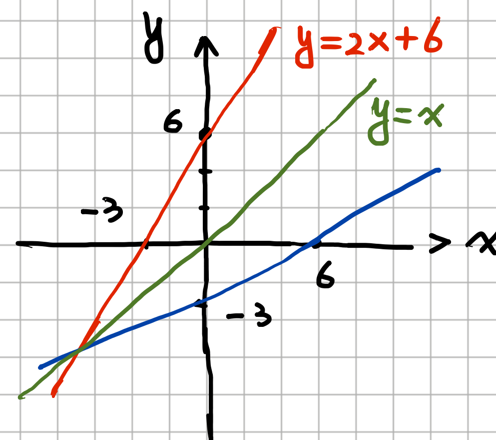
(2) #33 $f(x) = \frac{2}{x^2+1}$ for $x \geq 0$.

• let $y=\frac{2}{x^2+1} \Rightarrow x^2+1=\frac{2}{y} \Rightarrow x^2=\frac{2}{y}-1$

$\Rightarrow x=\pm\sqrt{\frac{2}{y}-1} \xrightarrow{x\geq 0} x=\sqrt{\frac{2}{y}-1} \Rightarrow y=\sqrt{\frac{2}{x}-1} \Rightarrow f^{-1}(x)=\sqrt{\frac{2}{x}-1}$

• Graphing inverse functions (27–32)

(a) $y=f(x)=2x+6$



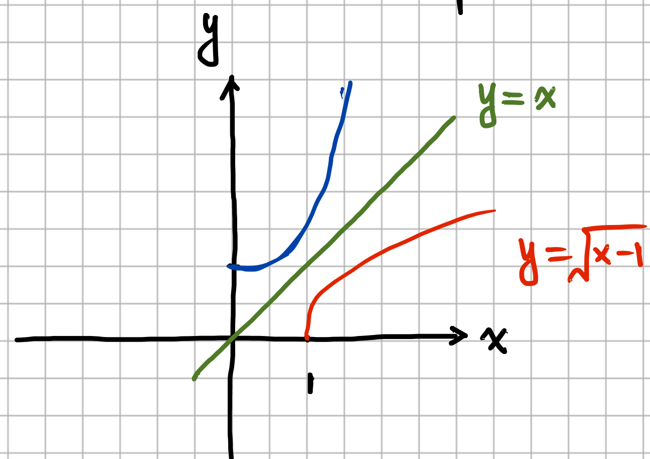
$y=f^{-1}(x)=\frac{x-6}{2}$

(b) $f(x)=\sqrt{x-1}$

$y=\sqrt{x-1} \Rightarrow y^2=x-1 \Rightarrow x=y^2+1$

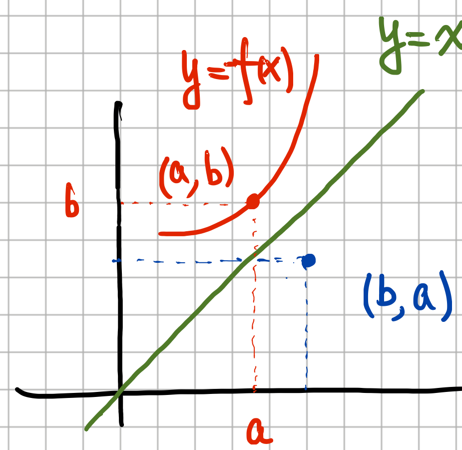
$\Rightarrow y=x^2+1$

$y=f^{-1}(x)=x^2+1$

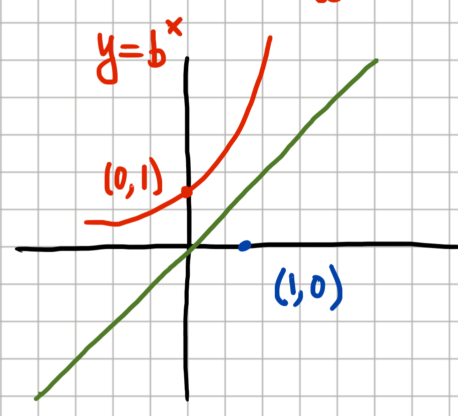


• $f(x)$ and $f^{-1}(x)$ are symmetric w.r.t. $y=x$

why?



• Logarithmic Functions



$y=\log_b x$

$y=\ln x$

- **Change-of-Base Rules**

$$b^x \overset{?}{=} e^{x \ln b} \qquad b^x = e^{\ln b^x} = e^{x \ln b}$$

$$\log_b x \overset{?}{=} \frac{\ln x}{\ln b}$$

set $y = \log_b x \implies b^y = b^{\log_b x} = x$

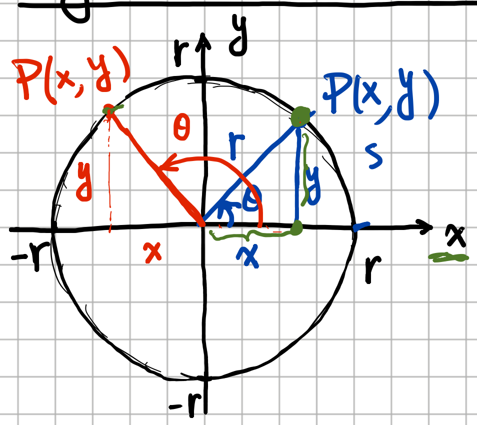
$\xrightarrow{\ln} \ln x = \ln b^y = y \ln b$

$\implies y = \dfrac{\ln x}{\ln b}$

§1.4 Trigonometric Functions and Their Inverses

(HW: 5-6, 14-17, 19-21, 23-24, 29, 35, 40, 51, 58, 65, 89, 97-98, 104, 106)

- **Trigonometric Functions**



$\sin\theta = \dfrac{y}{r}$ $\boxed{r=1} = y$

$\cos\theta = \dfrac{x}{r} = x$

$\tan\theta =$

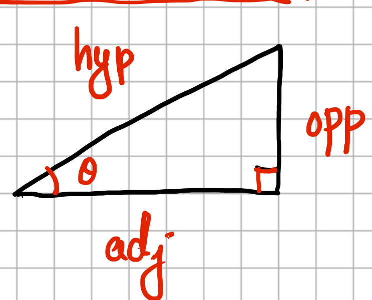
$\csc\theta =$

$\sec\theta =$

$\cot\theta =$

radian of $\theta = \dfrac{s}{r}$

$\boxed{0 \le \theta \le \frac{\pi}{2}}$



$\sin\theta = \text{opp}/\text{hyp}$

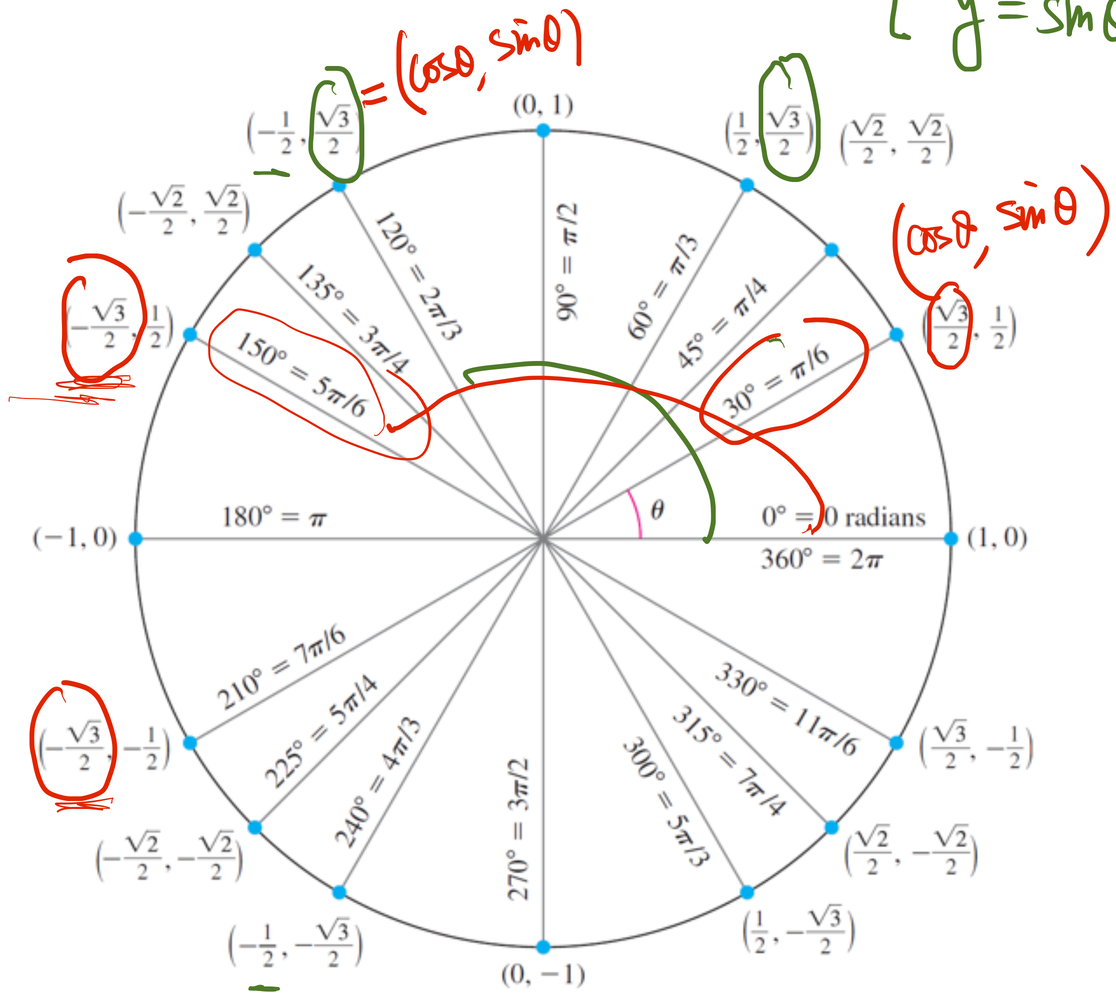
$\cos\theta =$

$\tan\theta =$

Figure 1.63

$$(x, y) = (\cos\theta, \sin\theta)$$

$$\Longrightarrow \begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$$



Evaluate $\sin\left(\frac{8\pi}{3}\right) =$

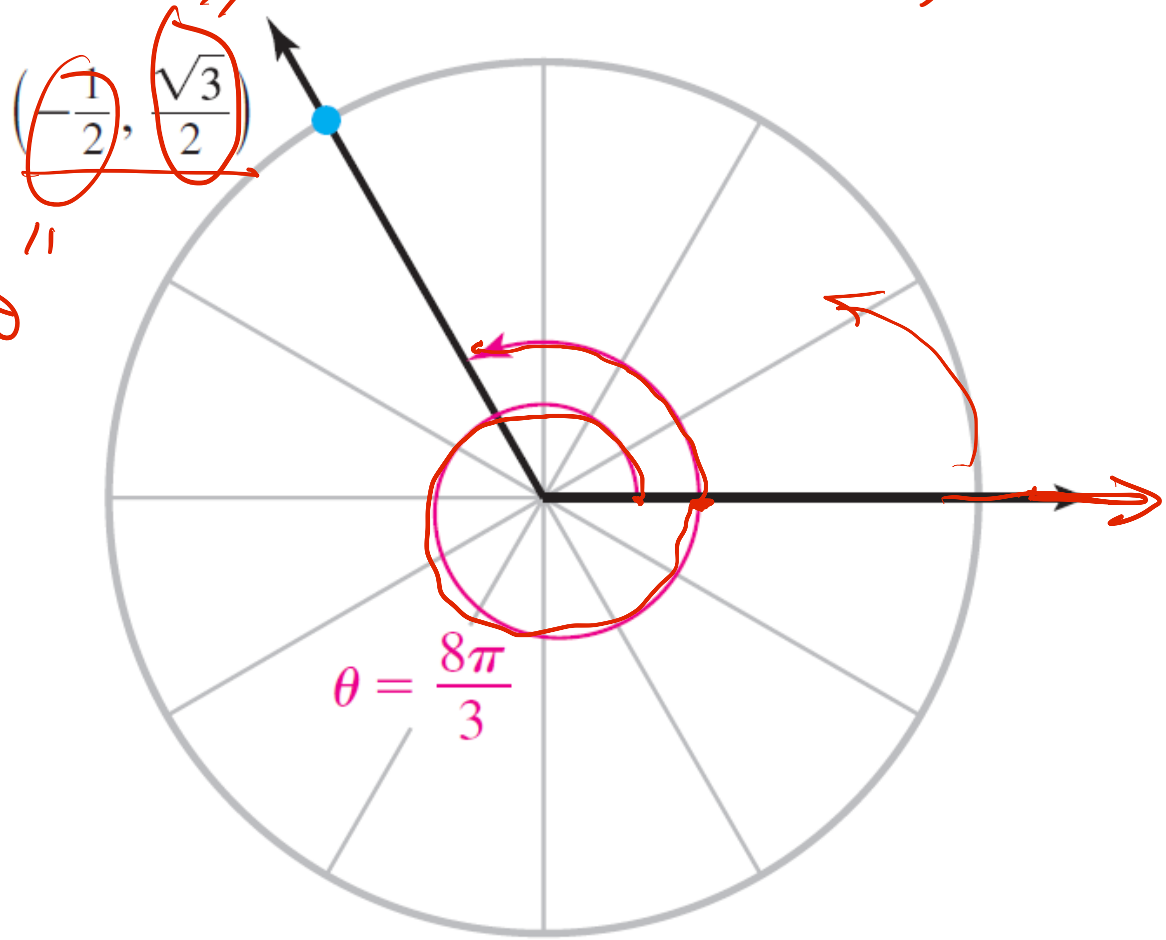
Figure 1.64

$$\frac{8\pi}{3} = 2\pi + \frac{2}{3}\pi$$

$$= \frac{6\pi + 2\pi}{3}$$

$= \sin\theta$

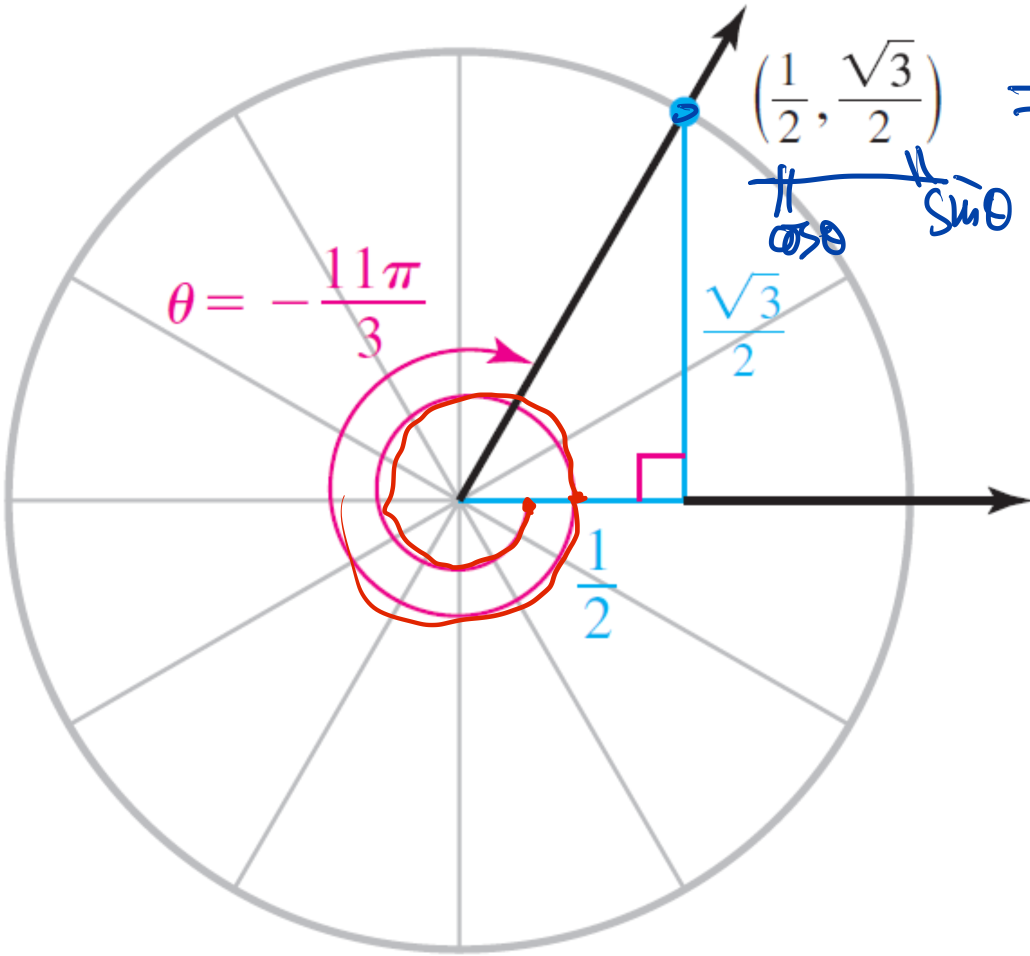
$= \cos\theta$



Evaluate $\csc\left(-\frac{11\pi}{3}\right) = \frac{1}{\sin(\ )} = \frac{2}{\sqrt{3}}$

Figure 1.65

$-\frac{11\pi}{3} = -2\pi - \frac{5\pi}{3} + \pi - \pi$



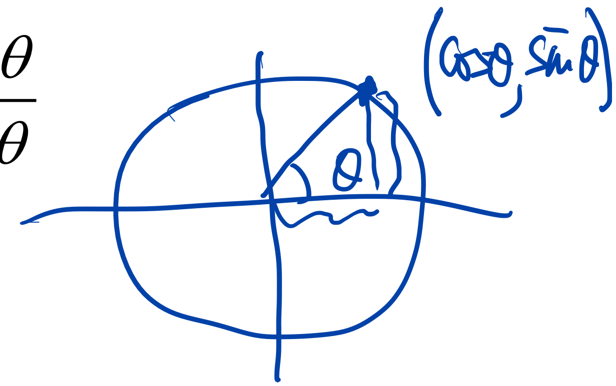
$= -4\pi + \frac{\pi}{3}$

$\cos\theta$ $\sin\theta$

Trigonometric Identities

Reciprocal Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$



$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta$$

Double- and Half-Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

Period of Trigonometric Functions

The functions $\sin\theta$, $\cos\theta$, $\sec\theta$, and $\csc\theta$ have a period of 2*π*:

$$\sin(\theta + 2\pi) = \sin\theta \qquad \cos(\theta + 2\pi) = \cos\theta$$

$$\sec(\theta + 2\pi) = \sec\theta \qquad \csc(\theta + 2\pi) = \csc\theta,$$

for all *θ* in the domain.

The functions $\tan\theta$ and $\cot\theta$ have a period of *π*:

$$\tan(\theta + \pi) = \tan\theta \qquad \cot(\theta + \pi) = \cot\theta$$

for all *θ* in the domain.

- Graphs

Figure 1.66 (a) & (b)

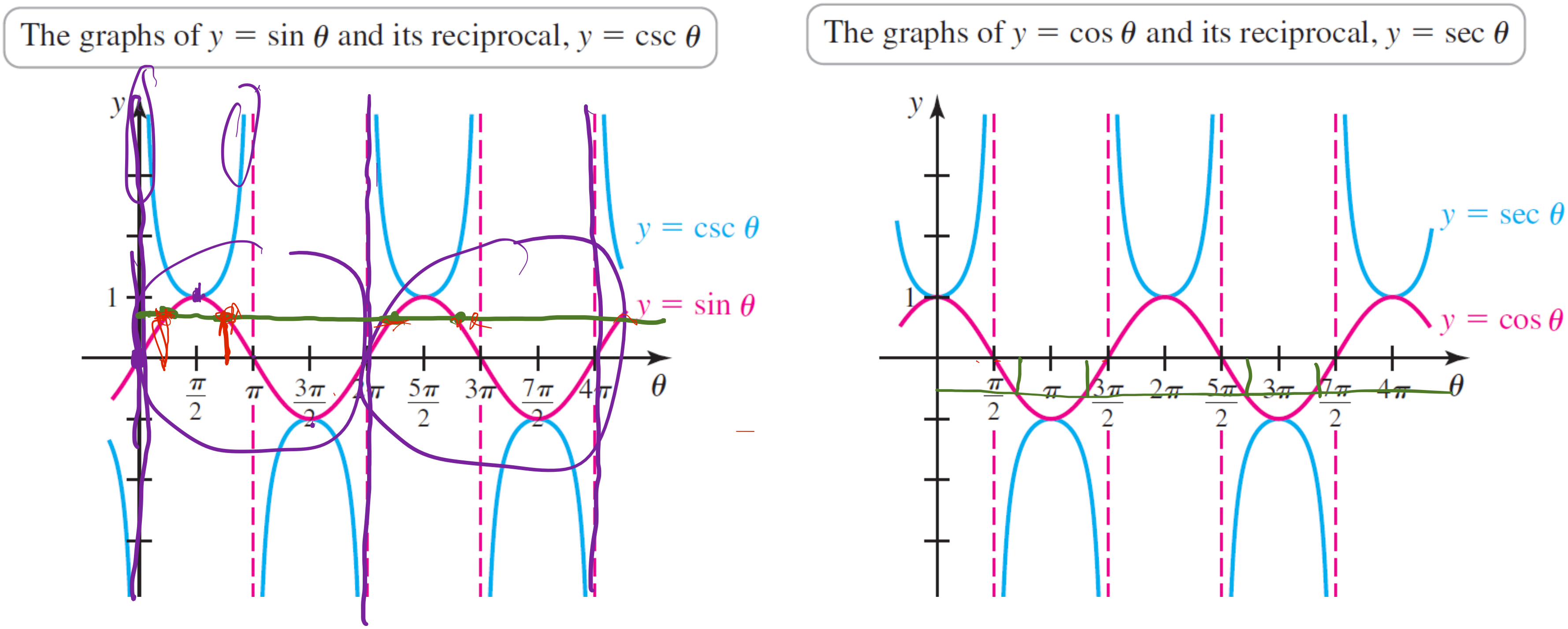
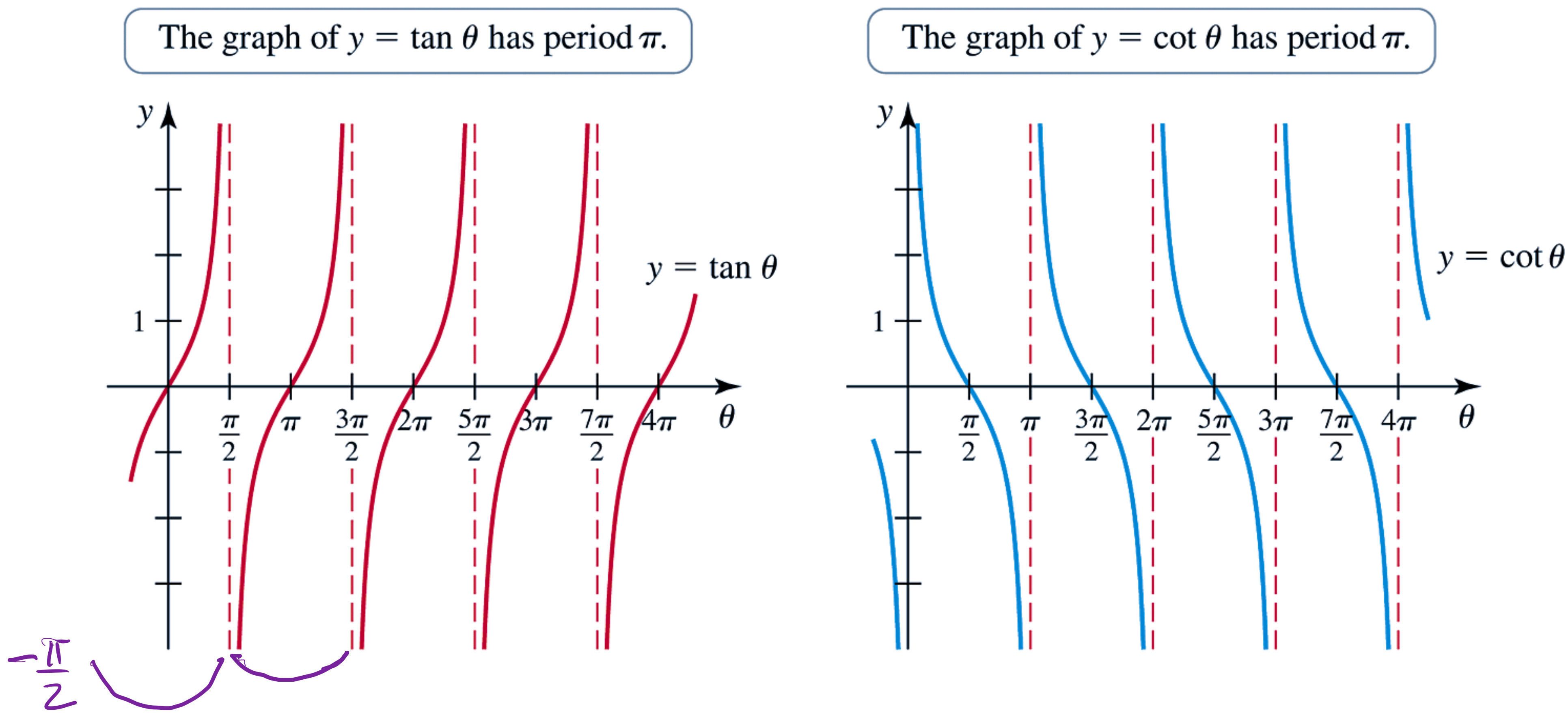


Figure 1.67 (a) & (b)



solving trigonometric equations (HW # 35, 40)

#39 $\sqrt{2}\sin x - 1 = 0$ $\qquad x = ?$

$$\sin x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad x = \frac{\pi}{4} \pm 2k\pi$$

$$\frac{3\pi}{4} \pm 2k\pi$$

#36 $2\theta\cos\theta + \theta = 0$

$$0 = \theta\,(2\cos\theta + 1) \Rightarrow \theta = 0 \text{ or } 2\cos\theta + 1 = 0$$

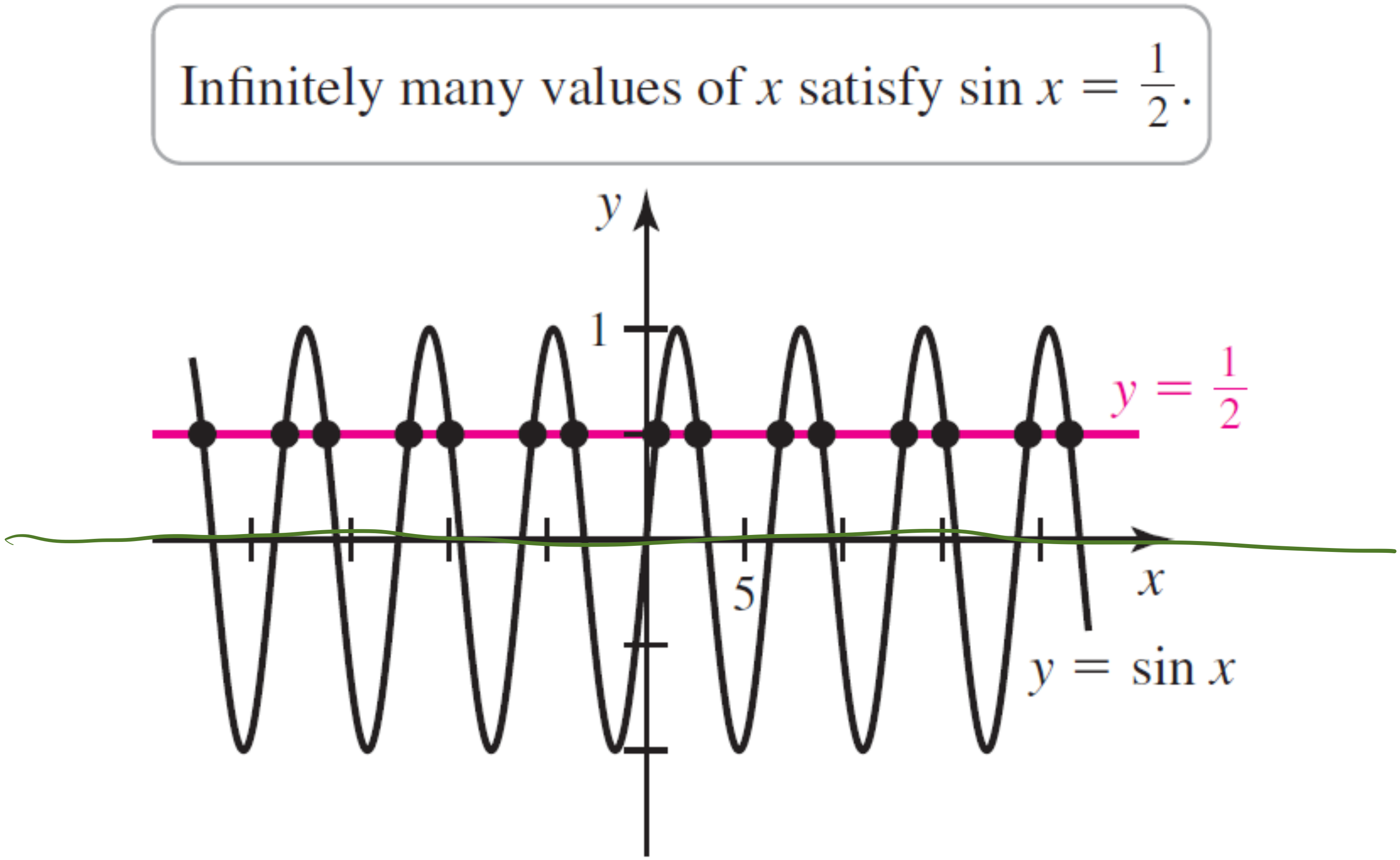
$$\Rightarrow \cos\theta = -\frac{1}{2}$$

$$\Rightarrow \theta = \frac{2\pi}{3} \pm 2k\pi \quad \text{for } k = 0, 1, \cdots$$

$$\frac{4\pi}{3} \pm 2k\pi$$

• Inverse of $\sin x$ and $\cos x$

Figure 1.70

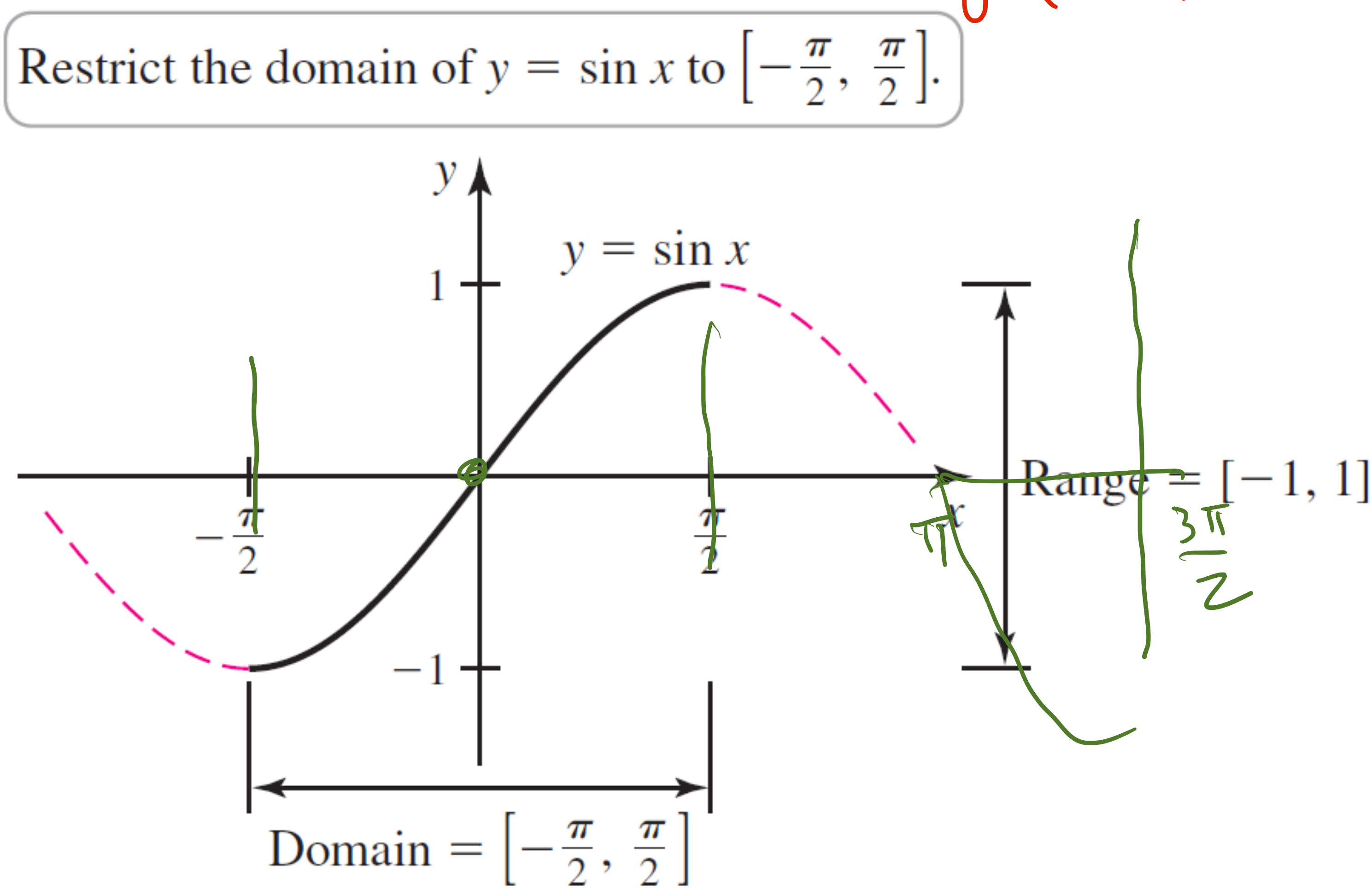


• $y = \sin^{-1} x \iff$

$\text{dom}(\sin^{-1} x) =$

$\text{range}(\sin^{-1} x) =$

Figure 1.71 (a)



$\bullet\ y = \cos^{-1} x \iff$

$\mathrm{dom}\left(\cos^{-1} x\right) =$

$\mathrm{range}\left(\cos^{-1} x\right) =$

Figure 1.71 (b)

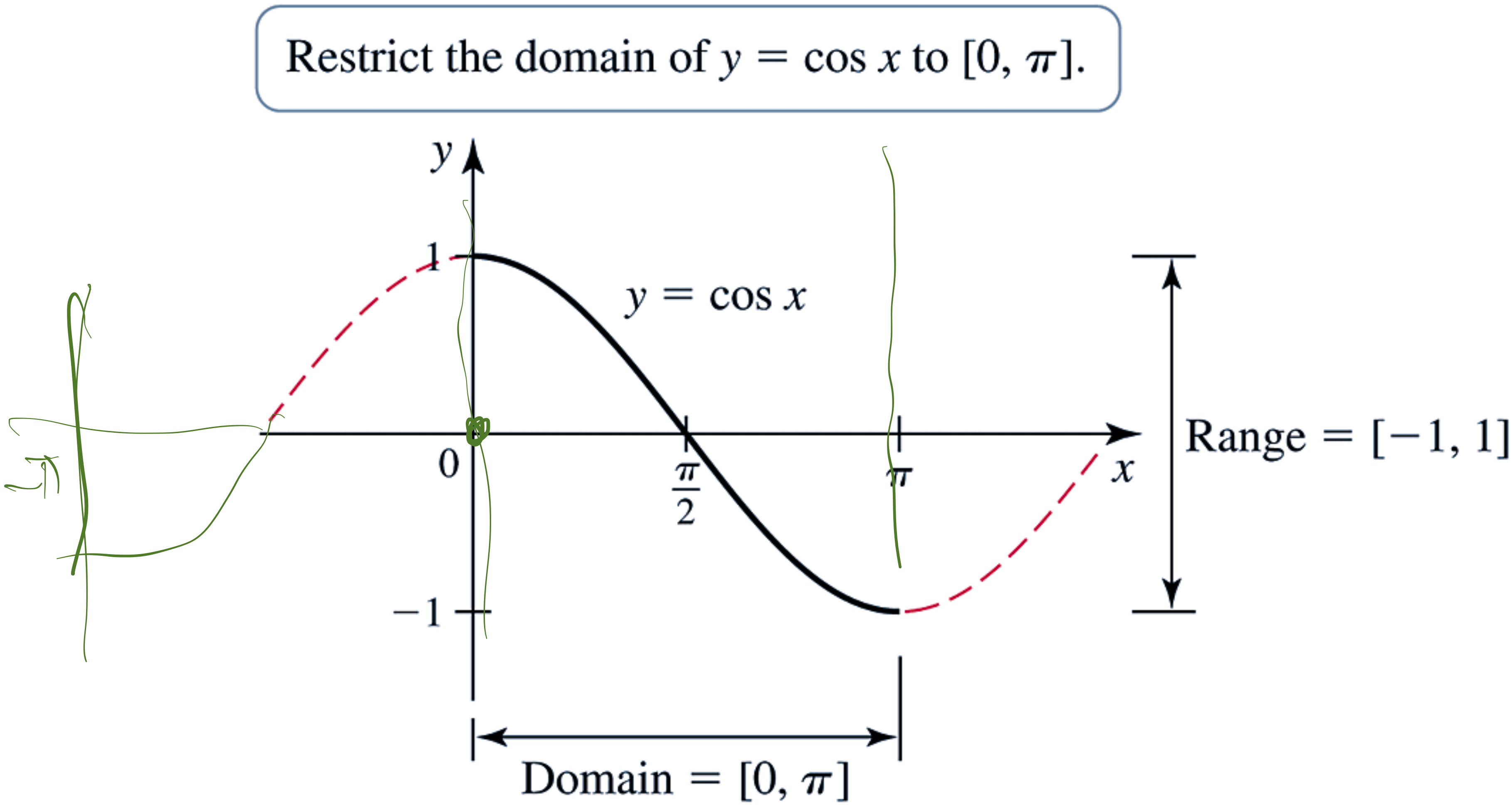
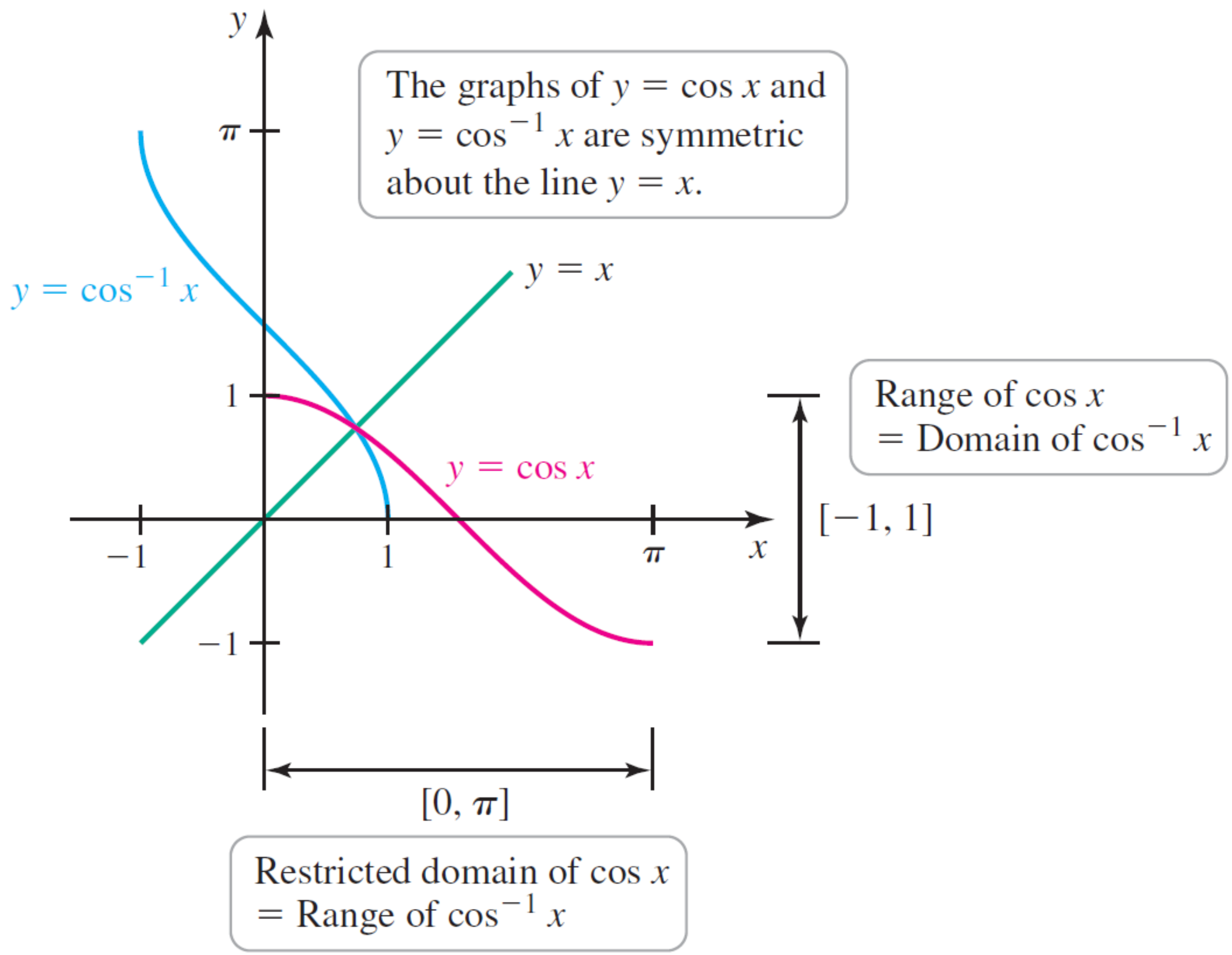


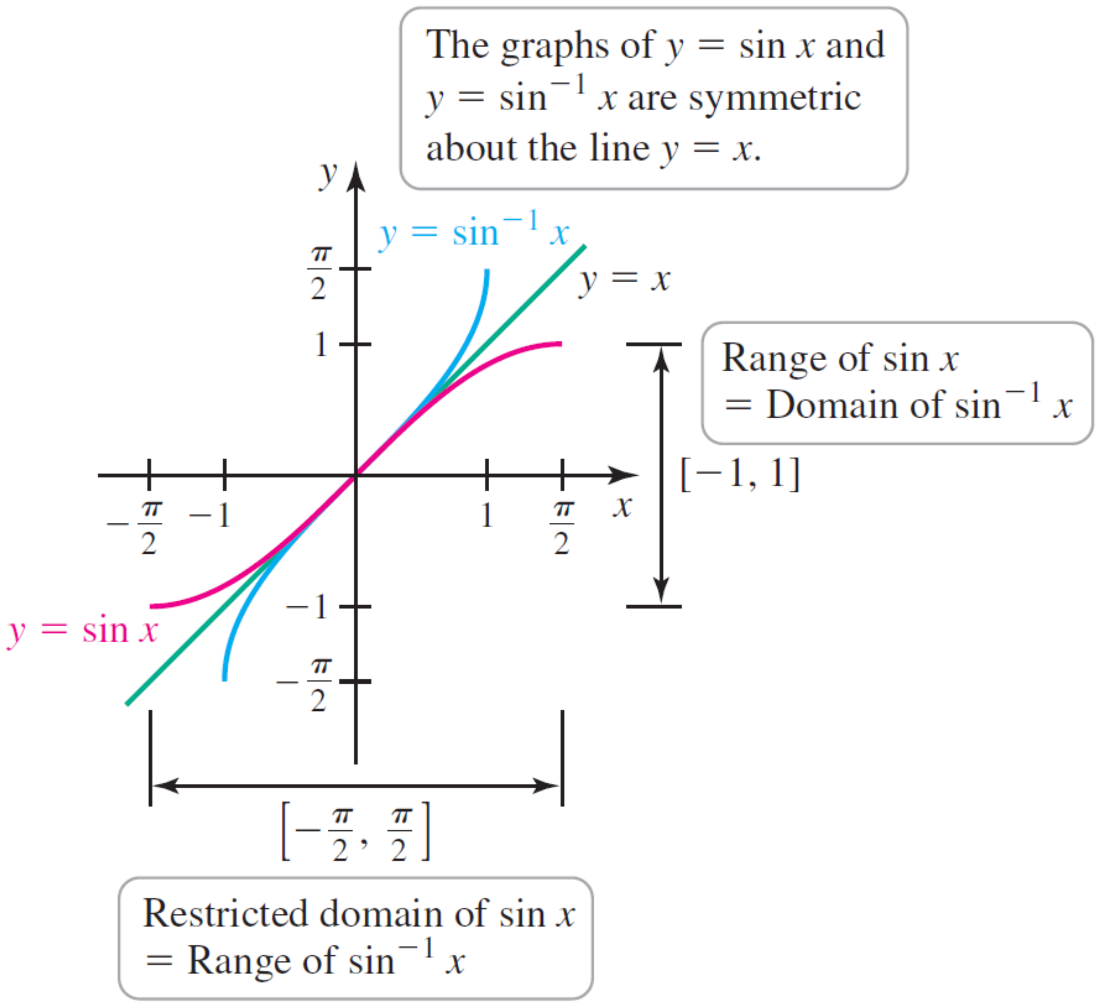
Figure 1.73



y
π
1
−1
−1
1
π
x
$y = \cos^{-1} x$
$y = x$
$y = \cos x$
The graphs of $y = \cos x$ and $y = \cos^{-1} x$ are symmetric about the line $y = x$.
Range of $\cos x$ $= $ Domain of $\cos^{-1} x$
$[-1, 1]$
$[0, \pi]$
Restricted domain of $\cos x$ $=$ Range of $\cos^{-1} x$

• Graphs

Figure 1.72



- Inverse Sine and Cosine

$$y = \sin^{-1} x \quad \text{and} \quad y = \cos^{-1} x$$

Examples

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta \iff \sin\theta = \frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{3}$$

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \theta \iff \cos\theta = -\frac{\sqrt{3}}{2} \implies \theta = \frac{5\pi}{6}$$

$$\cos^{-1}\left(\cos(3\pi)\right) \neq 3\pi \notin [0,\pi]$$

- $\cos(3\pi) = -1 \implies \theta = \cos^{-1}(-1)$

  $\iff \cos\theta = -1 \implies \theta = \pi$

or

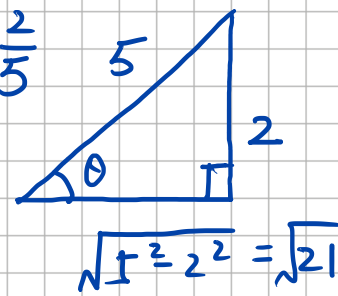
- $\cos(3\pi) = \cos(2\pi + \pi) = \cos\pi$

  $\implies \cos^{-1}(\cos 3\pi) = \cos^{-1}(\cos\pi) = \pi$

$$\sin^{-1}\left(\sin\left(\frac{3\pi}{4}\right)\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \theta \iff \sin\theta = \frac{\sqrt{2}}{2} \implies \theta = \frac{\pi}{4}$$

right-triangle relationships

- given $\theta = \sin^{-1}\left(\frac{2}{5}\right) \Leftrightarrow \sin\theta = \frac{2}{5}$

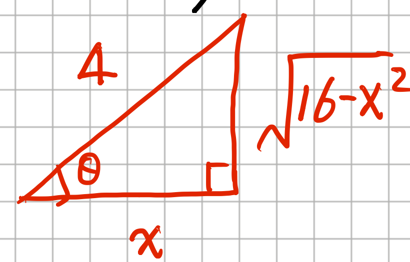


$\cos\theta = \frac{\sqrt{21}}{5}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2/5}{\sqrt{21}/5} = \frac{2}{\sqrt{21}}$.

- find an alternative form of $\cot\left(\cos^{-1}\left(\frac{x}{4}\right)\right)$ in term of x

Let $\theta = \cos^{-1}\left(\frac{x}{4}\right) \Longrightarrow \cos\theta = \frac{x}{4}$

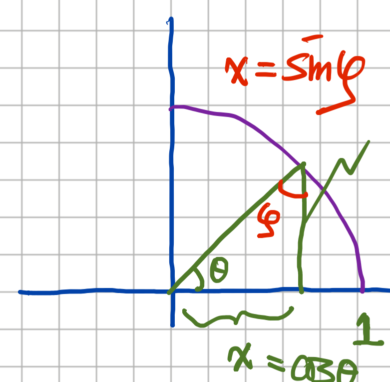


$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{x/4}{\sqrt{16-x^2}/4} = \frac{x}{\sqrt{16-x^2}}$

- $\cos^{-1}x + \sin^{-1}x \overset{?}{=} \frac{\pi}{2}$

Let $\theta = \cos^{-1}x \Longrightarrow \cos\theta = x$

$\varphi = \sin^{-1}x \qquad \sin\varphi = x$



$\Longrightarrow \frac{\pi}{2} = \theta + \varphi = \cos^{-1}x + \sin^{-1}x$.